Dynamical Systems
Tutorial 10: Smale Horseshoe

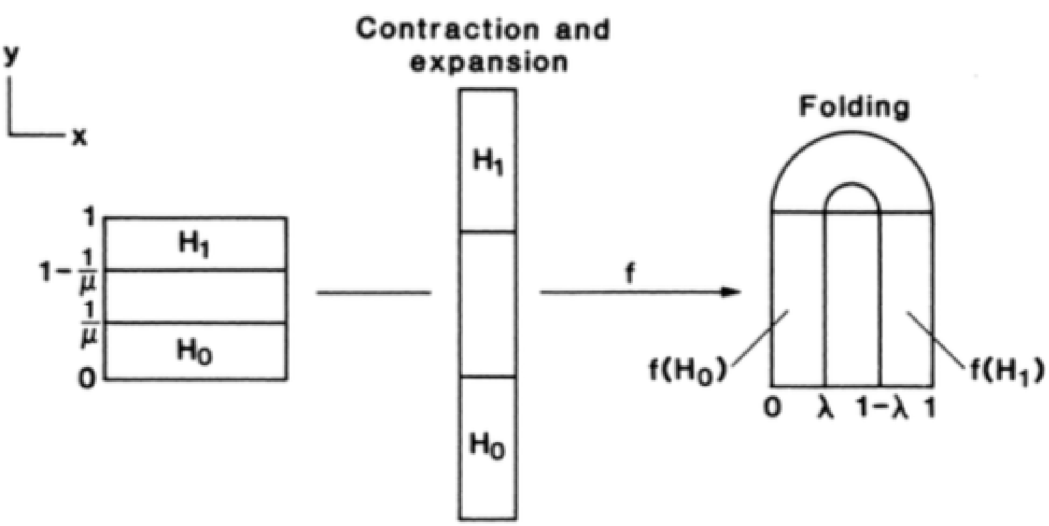
June 24, 2019

1 Definition of the Smale horseshoe map

Consider a map f, from the square D having sides of unit length into $\mathbb{R}^2$:

$$f : D \to \mathbb{R}^2, D = (x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1$$

which contracts the x-direction, expands the y-direction, and folds D around, laying it back on itself:



We will assume that f acts affinely on the "horizontal" rectangles

$$H_0 = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 1, 0 \leq y \leq 1/\mu\}, \tag{23.1.2}$$

and

$$H_1 = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 1, 1 - 1/\mu \leq y \leq 1\}, \tag{23.1.3}$$

taking them to the "vertical" rectangles

$$f(H_0) \equiv V_0 = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq \lambda, 0 \leq y \leq 1\}, \tag{23.1.4}$$

and

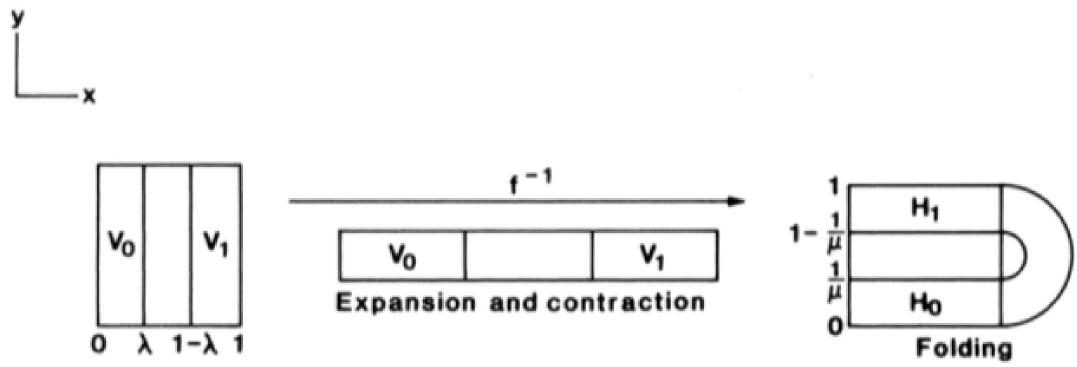
$$f(H_1) \equiv V_1 = \{(x, y) \in \mathbb{R}^2 \,|\, 1 - \lambda \leq x \leq 1, 0 \leq y \leq 1\}, \tag{23.1.5}$$

with the form of f on H_0 and H_1 given by

$$\begin{aligned} H_0 &: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \\ H_1 &: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -\lambda & 0 \\ 0 & -\mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ \mu \end{pmatrix}, \end{aligned} \tag{23.1.6}$$

and with $0 < \lambda < 1/2, \mu > 2$. (Note: the fact that on H_1 the matrix elements are negative means that in addition to being contracted in the x-direction by a factor λ and expanded in the y-direction by a factor μ, H_1 is also rotated 180°.)

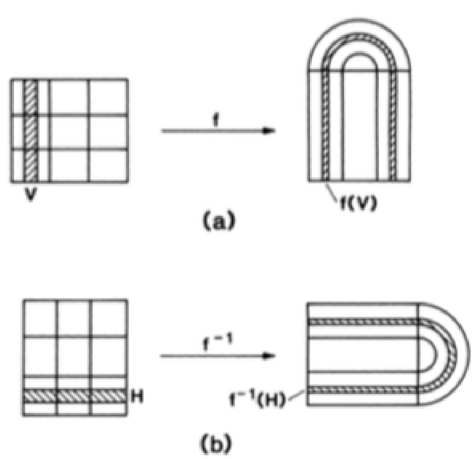
Additionally, it follows that f^{-1} acts on D as shown below, taking the "vertical" rectangles V_0 and V_1 to the "horizontal rectangles" H_0 and H_1 respectively.



We note the following property of f:

Lemma 23.1.1 a) *Suppose V is a vertical rectangle; then $f(V) \cap D$ consists of precisely two vertical rectangles, one in V_0 and one in V_1, with their widths each being equal to a factor of λ times the width of V.* b) *Suppose H is a horizontal rectangle; then $f^{-1}(H) \cap D$ consists of precisely two horizontal rectangles, one in H_0 and one in H_1, with their widths being a factor of $1/\mu$ times the width of H.*

By "vertical/horizontal rectangle" we mean a rectangle in D whose sides parallel to the y/x axis (respectively) have length one. The proof idea is as below, full proof may be found in Wiggins.



2 Construction of the invariant set

We will now geometrically construct the set of points Λ which remain in D under all possible iterations by f; thus Λ is defined by

$$\cdots \cap f^{-n}(D) \cap \cdots \cap f^{-1}(D) \cap D \cap f(D) \cap \cdots \cap f^{n}(D) \cap \cdots$$

or

$$\bigcap_{n=-\infty}^{\infty} f^{n}(D).$$

Denote $\Lambda^{+} = \bigcap_{n=0}^{\infty} f^{n}(D)$ and $\Lambda^{-} = \bigcap_{n=-\infty}^{0} f^{n}(D)$. We have $\Lambda = \Lambda^{+} \cap \Lambda^{-}$. We construct each of the invariant sets (forward and backward) separately, by induction, and then determine the limit. In order to keep track of the iterates of f at each step of the inductive process, let $S = \{0,1\}$ be an index set, and let s_i denote one of the two elements of S, i.e., $s_i \in S, i = 0, \pm 1, \pm 2, \cdots$.

2.1 The set Λ^+

$D \cap f(D)$. By the definition of f, $D \cap f(D)$ consists of the two vertical rectangles V_0 and V_1, which we denote as follows

$$D \cap f(D) = \bigcup_{s_{-1} \in S} V_{s_{-1}} = \{p \in D \,|\, p \in V_{s_{-1}},\ s_{-1} \in S\}, \qquad (23.2.1)$$

where $V_{s_{-1}}$ is a vertical rectangle of width λ; see Figure 23.2.1.

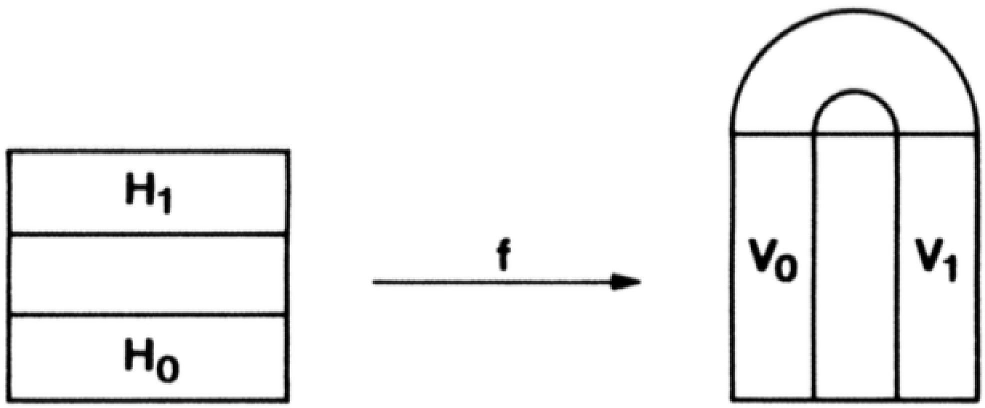


FIGURE 23.2.1.

$D \cap f(D) \cap f^2(D)$: Following some manipulations, it can be written as

$$\begin{aligned} &D \cap f(D) \cap f^2(D) \\ &\quad = \bigcup_{\substack{s_{-i} \in S \\ i=1,2}} (f(V_{s_{-2}}) \cap V_{s_{-1}}) \equiv \bigcup_{\substack{s_{-i} \in S \\ i=1,2}} V_{s_{-1}s_{-2}} \\ &\quad = \{p \in D \,|\, p \in V_{s_{-1}}, f^{-1}(p) \in V_{s_{-2}}, s_{-i} \in S, i = 1, 2\}. \quad (23.2.5) \end{aligned}$$

Pictorially, this set is described in Figure 23.2.2.

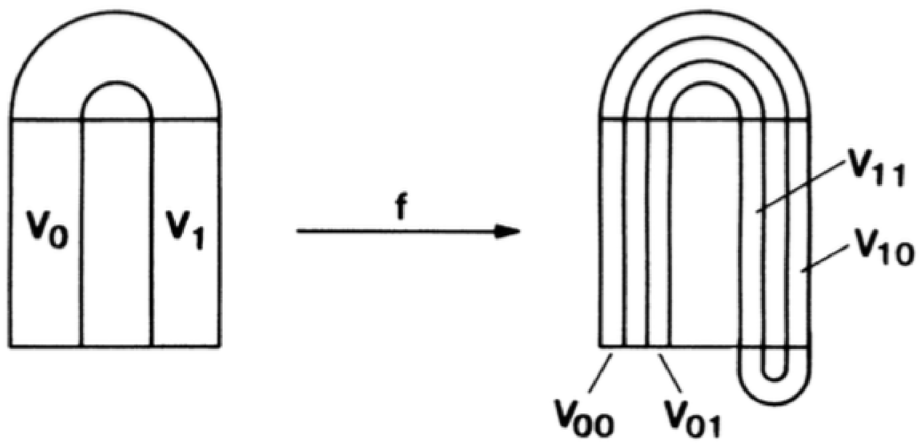
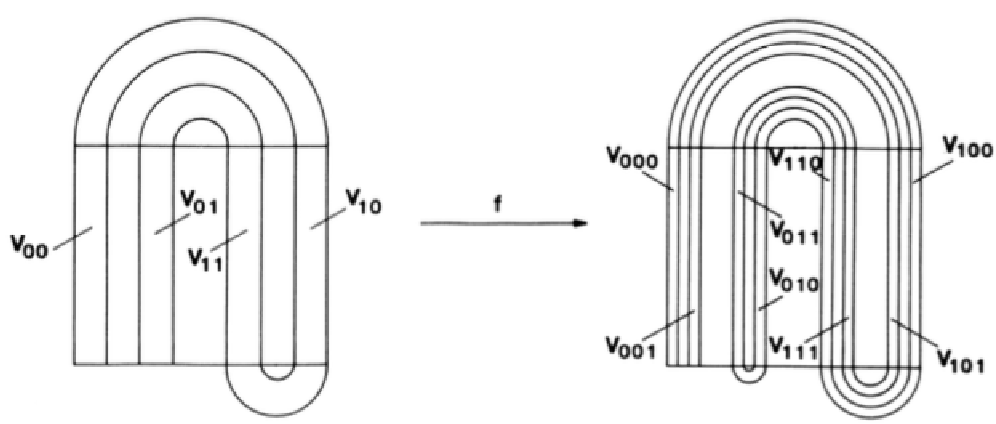


FIGURE 23.2.2.

So you can see we get $4 = 2^2$ vertical stripes of width λ^2.

$\underline{D \cap f(D) \cap f^2(D) \cap f^3(D)}$: Using the same reasoning as in the previous steps, this set consists of eight vertical rectangles, each having width λ^3:



which we denote as follows

$$
\begin{aligned}
&D \cap f(D) \cap f^2(D) \cap f^3(D) \\
&\quad = \bigcup_{\substack{s_{-1} \in S \\ i=1,2,3}} \left(f(V_{s_{-2}s_{-3}}) \cap V_{s_{-1}} \right) \equiv \bigcup_{\substack{s_{-i} \in S \\ i=1,2,3}} V_{s_{-1}s_{-2}s_{-3}} \\
&\quad = \big\{ p \in D \,|\, p \in V_{s-1}, f^{-1}(p) \in V_{s_{-2}}, \\
&\qquad\quad f^{-2}(p) \in V_{s_{-3}}, s_{-i} \in S, i = 1,2,3 \big\}, \qquad (23.2.6)
\end{aligned}
$$

Inductively, we can show that

$$
\begin{aligned}
&D \cap f(D) \cap \cdots \cap f^k(D) \\
&\quad = \bigcup_{\substack{s_{-i} \in S \\ i=1,2,\ldots,k}} \left(f(V_{s_{-2}\cdots s_{-k}}) \cap V_{s_{-1}} \right) \equiv \bigcup_{\substack{s_{-i} \in S \\ i=1,2,\ldots,k}} V_{s_{-1}\cdots s_{-k}} \\
&\quad = \big\{ p \in D \,|\, f^{-i+1}(p) \in V_{s_{-i}}, s_{-i} \in S, i = 1, \cdots, k \big\} \qquad (23.2.7)
\end{aligned}
$$

and that this set consists of 2^k vertical rectangles, each of width λ^k.

An important observation concerning the nature of this construction process: at the kth stage, we obtain 2^k vertical rectangles, and each vertical rectangle can be labeled by a sequence of 0's and 1's of length k. The important point to realize is that there are 2^k possible distinct sequences of 0's and 1's having length k and that each of these is realized in our construction process; thus, *the labeling of each vertical rectangle is unique at each step*. This fact follows from the geometric definition of f and the fact that V_0 and V_1 are disjoint. (This is unlike $f(x) = 2x$ mod 1.)

Inductively, we can show that

Letting $k \to \infty$, since a decreasing intersection of compact sets is non-empty, it is clear that we obtain an infinite number of vertical rectangles and that the width of each of these rectangles is zero, since $\lim_{k\to\infty} \lambda^k = 0$ for $0 < \lambda < 1/2$. Thus, we have shown that

$$\begin{aligned}\bigcap_{n=0}^{\infty} f^n(D) &= \bigcup_{\substack{s_{-i}\in S\\ i=1,2,\ldots}} \left(f(V_{s_{-2}\cdots s_{-k}\cdots}) \cap V_{s_{-1}}\right) \\ &\equiv \bigcup_{\substack{s_{-i}\in S\\ i=1,2,\ldots}} V_{s_{-1}\cdots s_{-k}\cdots} \\ &= \left\{p \in D \,|\, f^{-i+1}(p) \in V_{s_{-i}}, s_{-i} \in S, i = 1, 2, \cdots\right\}\end{aligned} \tag{23.2.8}$$

Each vertical line can be labeled by a unique infinite sequence of 0's and 1's as mentioned above.

2.2 The set Λ^-

In a similar manner, we have for the backward sets:

$D \cap f^{-1}(D)$. From the definition of f, this set consists of the two horizontal rectangles H_0 and H_1 and is denoted as follows

$$D \cap f^{-1}(D) = \bigcup_{s_0\in S} H_{s_0} = \left\{p \in D \,|\, p \in H_{s_0}, s_0 \in S\right\}. \tag{23.2.9}$$

See Figure 23.2.4.

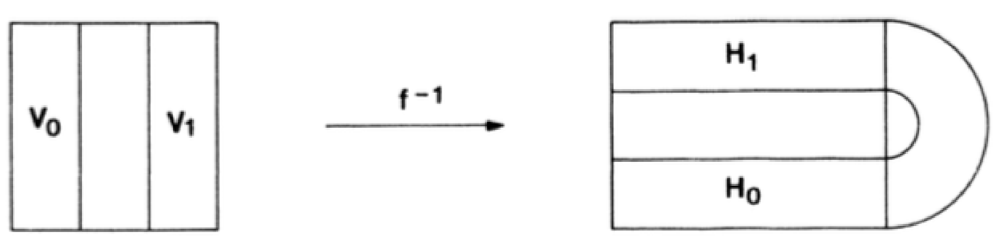
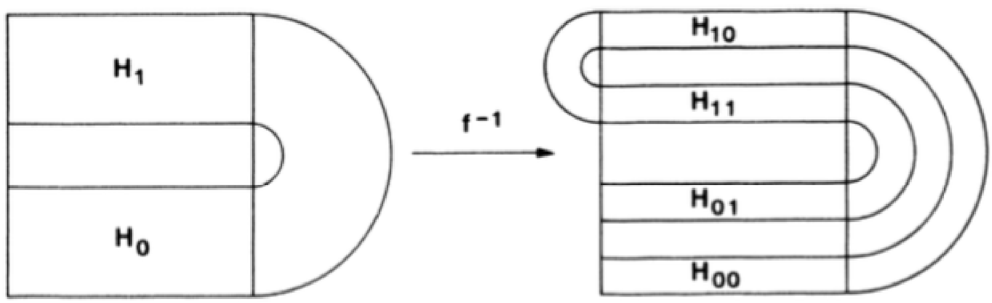


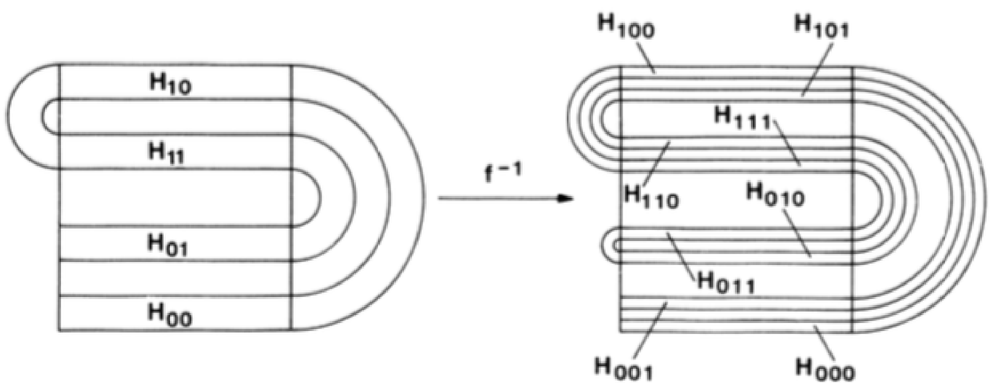
FIGURE 23.2.4.

$\underline{D \cap f^{-1}(D) \cap f^{-2}(D):}$



$$\begin{aligned} & D \cap f^{-1}(D) \cap f^{-2}(D) \\ & \quad = \bigcup_{\substack{s_i \in S \\ i=0,1}} \left(f^{-1}(H_{s_1}) \cap H_{s_0}\right) \equiv \bigcup_{\substack{s_i \in S \\ i=0,1}} H_{s_0 s_1} \\ & \quad = \left\{p \in D \,|\, p \in H_{s_0}, f(p) \in H_{s_1}, s_i \in S, i = 0,1\right\}. \qquad (23.2.12) \end{aligned}$$

$\underline{D \cap f^{-1}(D) \cap f^{-2}(D) \cap f^{-3}(D):}$



Continuing this procedure, at the kth step we obtain $D \cap f^{-1}(D) \cap \cdots \cap f^{-k}(D)$, which consists of 2^k horizontal rectangles each having width $1/\mu^k$, where again each rectangle is labeled uniquely by a sequence of 0's and 1's of length k.

$$
\begin{aligned}
&D \cap f^{-1}(D) \cap \cdots \cap f^{-k}(D) \\
&\quad = \bigcup_{\substack{s_i \in S \\ i=0,\cdots,k-1}} \left(f^{-1}(H_{s_1 \cdots s_{k-1}}) \cap H_{s_0}\right) \equiv \bigcup_{\substack{s_i \in S \\ i=0,\cdots,k-1}} H_{s_0 \cdots s_{k-1}} \\
&\quad = \left\{p \in D \,|\, f^i(p) \in H_{s_i}, s_i \in S, i = 0, \cdots, k-1\right\}. \qquad (23.2.14)
\end{aligned}
$$

Taking the limit $k \to \infty$, we get an infinite set of horizontal lines, each line labeled uniquely by 0's and 1's:

$$
\begin{aligned}
\bigcap_{-\infty}^{n=0} f^n(D) &= \bigcup_{\substack{s_i \in S \\ i=0,1,\cdots}} \left(f(H_{s_1 \cdots s_k \cdots}) \cap H_{s_0}\right) \equiv \bigcup_{\substack{s_i \in S \\ i=0,1,\cdots}} H_{s_0 \cdots s_k \cdots} \\
&= \left\{p \in D \,|\, f^i(p) \in H_{s_i}, s_i \in S, i = 0, 1, \cdots\right\}. \qquad (23.2.15)
\end{aligned}
$$

Thus, we have

$$\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(D) = \left[\bigcap_{n=-\infty}^{0} f^n(D)\right] \cap \left[\bigcap_{n=0}^{\infty} f^n(D)\right], \tag{23.2.16}$$

which consists of an infinite set of points, since each vertical line in $\bigcap_{n=0}^{\infty} f^n(D)$ intersects each horizontal line in $\bigcap_{-\infty}^{n=0} f^n(D)$ in a unique point. Furthermore, each point $p \in \Lambda$ can be labeled *uniquely* by a bi-infinite sequence of 0's and 1's which is obtained by concatenating the sequences associated with the respective vertical and horizontal lines that serve to define p. Stated more precisely, let $s_{-1} \cdots s_{-k} \cdots$ be a particular infinite sequence of 0's and 1's; then $V_{s_{-1} \cdots s_{-k} \cdots}$ corresponds to a unique vertical line. Let $s_0 \cdots s_k \cdots$ likewise be a particular infinite sequence of 0's and 1's; then $H_{s_0 \cdots s_k \cdots}$ corresponds to a unique horizontal line. Now a horizontal line and vertical line intersect in a unique point p; thus, we have a well-defined map from points $p \in \Lambda$ to bi-infinite sequences of 0's and 1's which we call ϕ.

$$p \overset{\phi}{\longmapsto} \cdots s_{-k} \cdots s_{-1} s_0 \cdots s_k \cdots .$$

Notice that because

$$\begin{aligned} V_{s_{-1} \cdots s_{-k} \cdots} &= \left\{p \in D \,|\, f^{-i+1}(p) \in V_{s_{-i}}, i = 1, \cdots\right\} \\ &= \left\{p \in D \,|\, f^{-i}(p) \in H_{s_{-i}}, i = 1, \cdots\right\} \\ &\quad \text{since } f(H_{s_i}) = V_{s_i} \end{aligned} \tag{23.2.17}$$

and

$$H_{s_0 \cdots s_k \cdots} = \left\{p \in D \,|\, f^i(p) \in H_{s_i}, i = 0, \cdots\right\}, \tag{23.2.18}$$

we have

$$\begin{aligned} p &= V_{s_{-1} \cdots s_{-k} \cdots} \cap H_{s_0 \cdots s_k \cdots} \\ &= \left\{p \in D \,|\, f^i(p) \in H_{s_i}, i = 0, \pm 1, \pm 2, \cdots\right\}. \end{aligned} \tag{23.2.19}$$

Therefore, we see that the unique sequence of 0's and 1's we have associated with p contains information concerning the behavior of p under iteration by f. In particular, the s_k-th element in the sequence associated with p indicates that $f^k(p) \in H_{s_k}$.

3 Symbolic Dynamics

Notice that for the bi-infinite sequence of 0's and 1's associated with any $p \in \Lambda$, the decimal point separates the past iterates from the future iterates; thus, the sequence of 0's and 1's associated with $f^k(p)$ is obtained from the sequence associated with p merely by shifting the decimal point in the sequence associated with p k places to the right if k is positive or k places to the left is k is negative. More accurately:

Theorem 5.1.1. *There is a 1–1 correspondence ϕ between Λ and the set Σ of bi-infinite sequences of two symbols such that the sequence $\mathbf{b} = \phi(f(x))$ is obtained from the sequence $\mathbf{a} = \phi(x)$ by shifting indices one place: $b_i = a_{i+1}$. The set Σ has a metric defined by*

$$d(a, b) = \sum_{i=-\infty}^{\infty} \delta_i 2^{-|i|}, \qquad \delta_i = \begin{cases} 0 & \text{if } a_i = b_i, \\ 1 & \text{if } a_i \neq b_i. \end{cases} \tag{5.1.2}$$

The map ϕ is a homeomorphism from Λ to Σ endowed with this metric.

You can find the proof at the end of the document or in Guckenheimer and Holmes.

The correspondence ϕ between Λ and Σ imparts to Λ a symbolic description which is an extraordinarily useful tool for understanding the dynamics of Λ. It is helpful to give a formal name to the process of "shifting indices." Thus

$$\sigma: \Sigma \to \Sigma, \tag{5.1.4}$$

the *shift map*, is defined by $\sigma(\mathbf{a}) = \mathbf{b}$ with $b_i = a_{i+1}$. The basic property of the theorem is now restated as the equation

$$\phi \circ (f|_\Lambda) = \sigma \circ \phi. \tag{5.1.5}$$

This equation expresses the topological conjugacy of $f|_\Lambda$ and σ. Written as $f|_\Lambda = \phi^{-1} \circ \sigma \circ \phi$, it has the immediate consequence that

$$f^n|_\Lambda = \phi^{-1} \circ \sigma^n \circ \phi, \tag{5.1.6}$$

so that ϕ maps orbits of f in Λ to orbits of σ in Σ. The description of σ is explicit enough that many dynamical properties are readily determined. For example, a periodic orbit of period n for σ consists of a sequence which is periodic: $a_i = a_{i+n}$ for all i in the sequence $\mathbf{a}$. Fixing n, we readily count the sequences with the property $a_i = a_{i+n}$ and find that f^n has 2^n fixed points in Λ. This set includes all points which are periodic with period n or a divisor of n.

To summarize, we have:

Theorem 5.1.2. *The horseshoe map f has an invariant Cantor set Λ such that:*

(a) *Λ contains a countable set of periodic orbits of arbitrarily long periods.*
(b) *Λ contains an uncountable set of bounded nonperiodic motions.*
(c) *Λ contains a dense orbit.*

Moreover, any sufficiently C^1 close map $\tilde{f}$ has an invariant Cantor set $\tilde{\Lambda}$ with $\tilde{f}|_{\tilde{\Lambda}}$ topologically equivalent to $f|_{\Lambda}$.

where the last part of the theorem is a result of the robustness of the horseshoe map - see more in (GH).

Bibliography

- Wiggins, S. (2003). Introduction to applied nonlinear dynamical systems and chaos.

- Holmes, J., and Guckenheimer, P. (2002). Nonlinear Oscillations Dynamical Systems and Bifurcations of Vector Fields.

Appendix - Proof of conjugacy between the shift map and the horseshoe map (GH)

Notice we used H_0 and H_1 whereas here they use H_1 and H_2, and we denoted the square D instead of S - everything is of course equivalent...

PROOF. The proof of this theorem provides a basic illustration of how symbolic dynamics works. Take the two symbols of the theorem to be 1 and 2. The map is defined by the recipe

$$\phi(x) = \{a_i\}_{i=-\infty}^{\infty}, \quad \text{with } f^i(x) \in H_{a_i}. \tag{5.1.3}$$

In words, x is in Λ if and only if $f^i(x)$ is in $H_1 \cup H_2$ for each i, and we associate to x the sequence of indices that tells us which of H_1 and H_2 contains each $f^i(x)$. Unlike the map $f = 2x \pmod 1$, this definition of ϕ is unambiguous because H_1 and H_2 are disjoint. This description of ϕ leads immediately to the shift property required by the theorem: since $f^{i+1}(x) = f^i(f(x))$, it follows that $\phi(f(x))$ is obtained from $\phi(x)$ by shifting indices. To see that ϕ is both 1–1 and continuous, we look at the set of x's which each possess a given central string of symbols. Specifying $b_{-m}, b_{-m+1}, \ldots \cdot b_0, \ldots, b_n$ we denote as $R(b_{-m}, b_{-m+1}, \ldots \cdot b_0, \ldots, b_n)$ the set of x's for which $f^i(x) \in H_{b_i}$ for $-m \leq i \leq n$. We observe inductively that $R(b_{-m}, \ldots, b_n)$ is a rectangle of height $\mu^{-(n+1)}$ and width λ^m, obtained from the intersection of a horizontal and a vertical strip. As one lets $m, n \to \infty$, the diameter of the sets $R(b_{-m}, \ldots, b_n) \to 0$. Consequently, ϕ is both 1–1 and continuous.

The final point is that ϕ is onto. This is crucial for the applications of symbolic dynamics. The reason that ϕ is onto is that for each choice of $b_{-m}, \ldots, b_n$, the set $R(b_{-m}, \ldots, b_n)$ is nonempty. To see this, reference to Figure 5.1.2 is helpful. Note that $R(b_0, \ldots, b_n)$ is a horizontal strip mapped vertically from top to bottom of the square S by f^{n+1}. Therefore, $f^{n+1}(R(b_0, \ldots, b_n))$ intersects each H_i and $R(b_0, \ldots, b_n, b_{n+1})$ is a nonempty horizontal strip extending across S. Similarly, we have already observed that $S \cap f(S) \cap \cdots \cap f^m(S)$ consists of 2^m vertical strips. Each of these is a set of the form $R(b_{-m}, \ldots, b_{-1})$ and all sequences $(b_{-m}, \ldots, b_{-1})$ occur. Finally, $R(b_{-m}, \ldots, b_n)$ is nonempty because every vertical strip $R(b_{-m}, \ldots, b_{-1})$ intersects every horizontal strip $R(b_0, \ldots, b_n)$ and $R(b_{-m}, \ldots, b_n) = R(b_{-m}, \ldots, b_{-1}) \cap R(b_0, \ldots, b_n)$. □

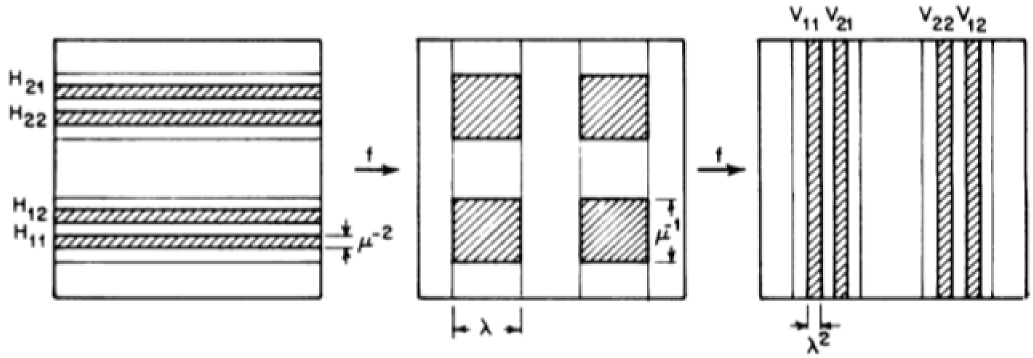


Figure 5.1.2. Iteration of f : $V_{ij} = f^2(H_{ij})$.